

MAS212 Linear Algebra I Exercises 6 (with solutions)

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Assessed submission deadline 5pm on Friday 17th November 2006 in the blue box on the ground floor.

Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together.

Marks out of 100 are shown in square brackets after each question number.

1. [20] For each of the following linear maps, write down the matrix A of the map with respect to the standard bases for the domain and codomain:

(a) $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}, \alpha(x_1, x_2) = 2x_1 - 3x_2;$

(b) $\alpha : \mathbb{F}_2^3 \rightarrow \mathbb{F}_2^3, \alpha(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3);$

(c) $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \alpha(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2);$

(d) $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \alpha(x_1, x_2) = (x_1 + x_2, x_1 - x_2, 0).$

Solution

Mapping the standard basis means setting each coordinate to 1 in turn, with all other coordinates set to 0, and the result is immediately a *column* of the matrix of the map, e.g. for (a) first column is $\alpha(1, 0) = 2$ and second column is $\alpha(0, 1) = -3$.

$$(a) A = \begin{bmatrix} 2 & -3 \end{bmatrix}, (b) A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, (c) A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, (d) A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Maple Solution

2. [30] State, with brief justification, which of the maps in Question 1 is an isomorphism and for each of these maps construct its inverse map. Hence write down the matrix B of the inverse map with respect to the standard bases and calculate the matrix products AB and BA .

Solution

For a map to be an isomorphism, the dimensions of its domain and codomain must be the same and hence the matrix must be square, which rules out (a) and (d). Claim that (b) and (c) are isomorphisms.

(b) Let $\alpha(x_1, x_2, x_3) = (y_1, y_2, y_3)$, so that $(x_1, x_2, x_3) = \alpha^{-1}(y_1, y_2, y_3)$. Solving

$\alpha(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3) = (y_1, y_2, y_3)$ for x_1, x_2, x_3 gives

$x_1 = y_1, x_2 = y_2 - y_1, x_3 = y_3 - y_2$ over $\mathbb{F}_2$. Hence $\alpha^{-1}(y_1, y_2, y_3) = (y_1, y_2 - y_1, y_3 - y_2)$. Therefore

$$B := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ from Q1 so } A.B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B.A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ over } \mathbb{F}_2.$$

(c) Let $\alpha(x_1, x_2, x_3) = (y_1, y_2, y_3)$, so that $(x_1, x_2, x_3) = \alpha^{-1}(y_1, y_2, y_3)$. Solving $\alpha(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2) = (y_1, y_2, y_3)$ for x_1, x_2, x_3 gives $x_1 = \frac{1}{2}(-y_1 + y_2 + y_3)$, $x_2 = \frac{1}{2}(y_1 - y_2 + y_3)$, $x_3 = \frac{1}{2}(y_1 + y_2 - y_3)$. Hence $\alpha^{-1}(y_1, y_2, y_3) = \frac{1}{2}(-y_1 + y_2 + y_3, y_1 - y_2 + y_3, y_1 + y_2 - y_3)$. Therefore

$$B := \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ from Q1 so } A.B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B.A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

► Maple Solution

3. [20] For each of the isomorphisms

(a) $\alpha: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \alpha(x_1, x_2) = (x_2, x_1)$,

(b) $\alpha: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \alpha(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 - x_3, x_2)$,

construct the inverse map α^{-1} , then write down the matrix A of the map α and the matrix B of the inverse map α^{-1} with respect to the standard bases for the domain and codomain and verify that $AB = BA = I$, where I is the identity matrix.

▼ Solution

(a) The effect of α is to reverse the coordinates so obviously α^{-1} must also reverse the coordinates in order to undo α . Therefore $\alpha^{-1}(x_1, x_2) = (x_2, x_1)$. For formally, solving $\alpha(x_1, x_2) = (x_2, x_1) = (y_1, y_2)$ for x_1, x_2 gives $x_1 = y_2, x_2 = y_1$, i.e. $\alpha^{-1}(y_1, y_2) = (x_1, x_2) = (y_2, y_1)$.

Hence $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Solving $\alpha(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 - x_3, x_2) = (y_1, y_2, y_3)$ for x_1, x_2, x_3 gives

$x_1 = -y_3 + \frac{1}{2}y_2 + \frac{1}{2}y_1, x_2 = y_3, x_3 = -y_3 - \frac{1}{2}y_2 + \frac{1}{2}y_1$, i.e.

$\alpha^{-1}(y_1, y_2, y_3) = \left(-y_3 + \frac{1}{2}y_2 + \frac{1}{2}y_1, y_3, -y_3 - \frac{1}{2}y_2 + \frac{1}{2}y_1\right)$.

Hence $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \frac{1}{2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 2 \\ 1 & -1 & -2 \end{bmatrix}$, so $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

► Maple Solution

4. [30] You may assume the rules of matrix transposition as given in the lecture notes.

(a) Write down a simple 2×3 matrix A with integer elements. Compute its transpose A^T . Choose a small integer k and compute $B = kAA^T$ and $C = kA^T A$. Show explicitly that $B^T = B$ and $C^T = C$. Prove that if A is any matrix over a field $\mathbb{K}$, $k \in \mathbb{K}$ and $B = kAA^T$ then $B^T = B$.

(b) Let $a, b, c, d, k \in \mathbb{K}$ and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Compute $B = k(A + A^T)$ and show explicitly that $B^T = B$. Prove that if A is *any square* matrix over a field $\mathbb{K}$, $k \in \mathbb{K}$ and $B = k(A + A^T)$ then $B^T = B$.

(c) Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$ be matrices over a field $\mathbb{K}$. Show explicitly that $(AB)^T = B^T A^T$ and $(BA)^T = A^T B^T$.

▼ **Solution**

(a) For example, let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \end{bmatrix}$ and let $k = 3$. Then $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$,

$$B = 3 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} = 3 \begin{bmatrix} 14 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 42 & -3 \\ -3 & 15 \end{bmatrix} \text{ and}$$

$$C = 3 \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \end{bmatrix} = 3 \begin{bmatrix} 5 & 2 & 1 \\ 2 & 4 & 6 \\ 1 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 6 & 3 \\ 6 & 12 & 18 \\ 3 & 18 & 21 \end{bmatrix}. \text{ So } B^T = \begin{bmatrix} 42 & -3 \\ -3 & 15 \end{bmatrix} = B \text{ and}$$

$$C^T = \begin{bmatrix} 15 & 6 & 3 \\ 6 & 12 & 18 \\ 3 & 18 & 21 \end{bmatrix} = C.$$

In general, if $B = kAA^T$ then $B^T = (kAA^T)^T = k(A^T)^T A^T = kAA^T = B$.

$$(b) A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow B = k(A + A^T) = k \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right) = \begin{bmatrix} 2ka & k(b+c) \\ k(b+c) & 2kd \end{bmatrix} = B^T.$$

In general, if $B = k(A + A^T)$ then $B^T = (k(A + A^T))^T = k(A^T + A) = B$.

$$(c) \text{ If } A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \text{ then } (AB)^T = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}^T = \begin{bmatrix} a_1 b_1 & a_2 b_1 \\ a_1 b_2 & a_2 b_2 \end{bmatrix} \text{ and}$$

$$B^T A^T = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_2 b_1 \\ a_1 b_2 & a_2 b_2 \end{bmatrix} \text{ so } (AB)^T = B^T A^T;$$

$(BA)^T = (a_1 b_1 + a_2 b_2)^T = a_1 b_1 + a_2 b_2$ since a scalar of 1×1 matrix transposes to itself and

$$A^T B^T = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2 \text{ so } (BA)^T = A^T B^T.$$

► **Maple Solution**